Example Context Elaboration: Dice to 100

Focus: Expected value

Achievement objective S8-4

In a range of meaningful contexts, students will be engaged in thinking mathematically and statistically. They will solve problems and model situations that require them to:

Investigate situations that involve elements of chance:

A calculating probabilities of independent, combined, and conditional events

B calculating and interpreting expected values and standard deviations of discrete random variables

C applying distributions such as the Poisson, binomial, and normal

Expected Value of a Dice Game

Laura and Bethany each throw a fair die, recording the score shown. The first player to reach a total of 100 wins. Before they begin to play they estimate how many turns it will take.

When they have played several games they realize that the number of games depends on the mean score for each toss of the die. To calculate this mean score they reason that they need to throw the die a large number of times (or use a simulation). The possible results for throwing the die 120 times are shown in the table.

Score on upper face	1	2	3	4	5	6	
frequency	21	19	24	13	17	26	120

From these results they calculate the mean score :

$$\bar{x} = \frac{\sum x.f}{\sum f} = \frac{(1 \text{ x } 21) + (2 \text{ x } 19) + (3 \text{ x } 24) + (4 \text{ x } 13) + (5 \text{ x } 17) + (6 \text{ x } 26)}{120} = \frac{424}{120} = 3.533$$

Their teacher discusses with Laura and Bethany that what they are dealing with here is a random process, the outcome of which cannot be predicted in advance. The value that shows on the die is called a random variable, usually denoted by an uppercase X. Although each individual outcome of tossing a die cannot be predicted it is possible to assign a probability to each possible outcome. This is called the probability distribution function of X. For this situation the probability distribution function is

x	1	2	3	4	5	6
P(X = x)	1/6	1/6	1/6	1/6	1/6	1/6

Students draw a graph for this probability distribution and comment on what they notice. Their teacher encourages them to think about what would have happened if their experiment had worked out perfectly. They realize that in that case they would have expected a roughly equal number of each score.

Score on upper face	1	2	3	4	5	6	
Experimental frequency	21	19	24	13	17	26	120
Theoretical frequency	20	20	20	20	20	20	120

Students draw a graph of these two distributions, comment on what they notice and visually predict the location of the mean.

In this case the calculation of the mean score for each toss of the die is

$$\bar{x} = \frac{\sum x.f}{\sum f} = \frac{(1 \times 20) + (2 \times 20) + (3 \times 20) + (4 \times 20) + (5 \times 20) + (6 \times 20)}{120} = \frac{420}{120} = 3.5$$

This leads Laura and Bethany to conclude that to reach a total of 100 points a player would have to throw the die about 100 / 3.5 times, that is about 29 times.

Laura and Bethany's teacher explains that this value, 3.5, is called the **expected value** of the score on the die, also called the mean, denoted by μ and explains that the formula to calculate it could be written like this:

$$\bar{x} = \frac{\sum x.f}{\sum f} = \frac{(1 \times 20) + (2 \times 20) + (3 \times 20) + (4 \times 20) + (5 \times 20) + (6 \times 20)}{120} = \frac{420}{120} = 3.5$$

To work towards E(X) formula…

Splitting this into separate fractions:

$$\frac{(1 \times 20)}{120} + \frac{(2 \times 20)}{120} + \frac{(3 \times 20)}{120} + \frac{(4 \times 20)}{120} + \frac{(5 \times 20)}{120} + \frac{(6 \times 20)}{120} = \frac{420}{120} = 3.5$$

This can be simplified as follows:

$$(1 \times \frac{20)}{120} + (2 \times \frac{20)}{120} + (3 \times \frac{20)}{120} + (4 \times \frac{20)}{120} + (5 \times \frac{20)}{120} + (6 \times \frac{20)}{120} = \frac{420}{120} = 3.5$$

which can be further simplified to

$\left(1 \times \frac{1}{6}\right) + \left(2 \times \frac{1}{6}\right) + \left(3 \times \frac{1}{6}\right) + \left(4 \times \frac{1}{6}\right) + \left(5 \times \frac{1}{6}\right) + \left(6 \times \frac{1}{6}\right) = \quad = \quad 3.5$

Laura and Bethany notice that that this is equivalent to multiplying each of the possible outcomes 1 to 6 by the probability of that outcome occurring and finding the total, that is

$$E(X) = \mu = \sum x \cdot P(X = x) \text{ for all values of } x.$$